MATHEMATICS

B.Sc. Part II (Paper III)

Topic : Further test of convergence

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Test the convergence of the series

$$\frac{x}{2}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{x^3}{6}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\cdot\frac{x^5}{10}+\ldots\text{to }\infty.$$

Solution : Let the n^{th} term of the given series be denoted by u_n.

Then $u_n=\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\ldots\ldots\frac{4n-7}{4n-6}\cdot\frac{4n-5}{4n-4}\cdot\frac{x^{2n-1}}{4n-2}$.

Replacing n by $(n+1)$, we get

$$u_{n+1}=\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\ldots\frac{4n-5}{4n-4}\cdot\frac{4n-3}{4n-2}\cdot\frac{4n-1}{4n}\cdot\frac{x^{2n+1}}{4n+2}$$

Then $\frac{u_{n+1}}{u_n}=\frac{4n-3}{4n-2}\cdot\frac{4n-1}{4n}\cdot\frac{4n-2}{4n+2}\cdot x^2$

$$=\frac{4-\frac{3}{n}}{4}\cdot\frac{4-\frac{1}{n}}{4+\frac{2}{n}}\cdot x^2.$$

$$\therefore \lim_{n\to\infty}\frac{u_{n+1}}{u_n}=x^2.$$

Hence, by D'Alembert's ratio test, the given series is convergent or divergent according as $x^2<1$ or $x^2>1$, that is, according as $|x|<1$ or $|x|>1$.

when $x^2=1$, then this test fails. Now Raabe's test can be applied.

$$\frac{u_n}{u_{n+1}}=\frac{4n(4n+2)}{(4n-3)(4n-1)}$$

or $\frac{u_n}{u_{n+1}}-1=\frac{4n(4n+2)}{(4n-3)(4n-1)}-1$

$$= \frac{16n^2+8n-16n^2+16n-3}{(4n-3)(4n-1)} = \frac{24n-3}{(4n-3)(4n-1)}$$

or $n\left(\frac{u_n}{u_{n+1}}-1\right) = \frac{24n^2-3n}{(4n-3)(4n-1)} = \frac{24-\frac{3}{n}}{\left(4-\frac{3}{n}\right)\left(4-\frac{1}{n}\right)}$

or $\lim_{n\to\infty}\left[n\left(\frac{u_n}{u_{n+1}}-1\right)\right] = \frac{24}{16} = \frac{3}{2} > 1$.

Hence, by Raabe's test, the given series is convergent, when $n^2 = 1$, that is, $|n| = 1$.

Example II Examine the convergence of the following series:

$$\sum \frac{(n+1)(n+2)}{(n+3)(n+4)}$$

Solution: Let the n^{th} term of the given series be denoted by u_n.

Then
$$u_n = \frac{(n+1)(n+2)}{(n+3)(n+4)}.$$

Replacing n by $(n+1)$, we get $u_{n+1} = \frac{(n+2)(n+3)}{(n+4)(n+5)}$.

$$\therefore \frac{u_{n+1}}{u_n} = \frac{(n+3)^2}{(n+1)(n+5)} = \frac{\left(1+\frac{3}{n}\right)^2}{\left(1+\frac{1}{n}\right)\left(1+\frac{5}{n}\right)}$$

or
$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = 1.$$

Hence D'Alembert's ratio test fails. Now Raabe's test can be applied.

$$\frac{u_n}{u_{n+1}} = \frac{(n+1)(n+5)}{(n+3)^2}$$

or $\frac{u_n}{u_{n+1}} - 1 = \frac{n^2+6n+5}{(n+3)^2} - 1 = \frac{n^2+6n+5-(n^2+6n+9)}{(n+3)^2}$.

or
$$n\left(\frac{u_n}{u_{n+1}}-1\right) = \frac{-4n}{(n+3)^2} = \frac{-\frac{4}{n}}{\left(1+\frac{3}{n}\right)^2};$$

or
$$\lim_{n\to\infty}\left[\left(\frac{u_n}{u_{n+1}}-1\right)\right] = 0 < 1.$$

Hence by Raabe's test, the given series is divergent.

Example III Test the convergence for positive real value of x:

$$x^2(\log_e 2)^p + x^3(\log_e 3)^p + x^4(\log_e 4)^p + \ldots \text{ to } \infty.$$

Solution

Let the nth term of the given series be denoted by u_n.

Then $u_n = x^{n+1}\{\log(n+1)\}^p$ and $u_{n+1} = x^{n+2}\{\log(n+2)\}^p$.

$$\therefore \frac{u_n}{u_{n+1}} = \left\{\frac{\log(n+1)}{\log(n+2)}\right\}^p \cdot \frac{1}{x} = \left\{\frac{\log(n+2-1)}{\log(n+2)}\right\}^p \cdot \frac{1}{x}$$

$$= \left[\frac{\log\left\{(n+2)\left(1-\frac{1}{n+2}\right)\right\}}{\log(n+2)}\right]^p \cdot \frac{1}{x}$$

$$= \left[\frac{\log(n+2) + \log\left(1+\frac{1}{n+2}\right)}{\log(n+2)}\right]^p \cdot \frac{1}{x}$$

$$= \left[1 - \frac{1}{(n+2)\log(n+2)} - \frac{1}{2(n+2)^2\log(n+2)} - \ldots\right]^p \cdot \frac{1}{x}$$

$$= \left[1 - \frac{p}{(n+2)\log(n+2)} - \frac{p}{2(n+2)^2\log(n+2)} - \ldots\right] \cdot \frac{1}{x}$$

or $$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{x}.$$

Hence, by D'Alembert's ratio test, the given series is convergent or divergent according as $x<1$ or $x>1$.

When $x=1$, then this test fails.

Now Raabe's test can be applied.

$$\frac{u_n}{u_{n+1}} - 1 = -\frac{p}{(n+2)\log(n+2)} - \frac{p}{2(n+2)^2\log(n+2)} - \ldots \text{ to } \infty$$

or $n\left(\frac{u_n}{u_{n+1}} - 1\right)$

$$= -\frac{p}{\left(1+\frac{2}{n}\right)\log(n+2)} - \frac{\frac{p}{n}}{2\left(1+\frac{2}{n}\right)^2\log(n+2)} - \ldots \text{ to } \infty$$

or $$\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)\right\}=0<1.$$

Hence, by Raabe's test, the given series is divergent when $x=1$.

EXAMPLE IV Test the convergence of the series whose nth term u_n is given by

$$u_n=\frac{2^2\cdot4^2\cdot6^2\cdots\cdots(2n-2)^2}{3\cdot4\cdot5\cdots\cdots(2n-2)(2n-1)}x^{2n}.$$

Solution : We have $u_n=\dfrac{2^2 4^2 6^2\cdots\cdots(2n-2)^2}{3\cdot4\cdot5\cdots(2n-2)(2n-1)}x^{2n}$.

Replacing n by $(n+1)$, we get

$$u_{n+1}=\frac{2^2\cdot4^2\cdot6^2\cdots\cdots(2n-2)^2(2n)^2}{3\cdot4\cdot5\cdots(2n-1)(2n)(2n+1)}x^{2n+2}$$

$$\therefore\ \frac{u_{n+1}}{u_n}=\frac{(2n)}{2n+1}x^2=\frac{1}{1+\frac{1}{2n}}\cdot x^2$$

or $$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=x^2.$$

Hence, by D'Alembert's ratio test, the given series is convergent or divergent according as $x^2<1$ or $x^2>1$. when $x^2=1$, then this test fails. Now Raabe's test can be applied.

$$\frac{u_n}{u_{n+1}}=\frac{2n+1}{2n};\ \text{or}\ \frac{u_n}{u_{n+1}}-1=\frac{2n+1}{2n}-1=\frac{1}{2n}$$

or $$n\left(\frac{u_n}{u_{n+1}}-1\right)=\frac{1}{2};$$

or $$\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)\right\}=\frac{1}{2}<1.$$

Hence, by Raabe's test the given series is divergent.